BU-329-M Federer

An augmented design is any standard experiment design which has been enlarged by additional treatments. The treatments in the standard experiment design are termed standard treatments, and the additional treatments are termed augmented or new treatments. The standard treatments are replicated r times as in the standard experiment design but the augmented treatments may occur 1, 2, ..., t times. In a previous paper the author gave a general statistical treatment of analysis of augmented designs with one-way elimination of heterogeneity in the experimental area or material. In the present paper general analyses for two-way elimination of heterogeneity for augmented latin squares and rectangles and for three-way elimination of heterogeneity for such augmented designs as the latin cube, the magic latin square, and the lattice square designs are treated in detail. The statistical analysis of the data for the standard treatments only follows that for the standard designs.

The construction of plans and the randomization procedures are given also. Both intrablock and interblock analyses are given. Designs with four-way and higher-way elimination of heterogeneity are briefly discussed.

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AUGMENTED DESIGNS WITH TWO-, THREE- AND HIGHER-WAY ELIMINATION OF HETEROGENEITY[1]

Walter T. Federer, Cornell University

I. Introduction

A general treatment of augmented designs with one-way elimination of heterogeneity has been presented [Federer, 1960]. The purpose of this paper is to present general analyses for augmented designs with two-, three-, and higher-way elimination of heterogeneity in the experimental area. To reiterate, an augmented design is simply any standard experimental design which has been enlarged by adding additional treatments. The treatments added are denoted as new treatments while the treatments occurring in the standard experimental design are denoted as standards or standard treatments. The new treatments could also have been denoted as the augmented treatments.

Experimental designs with two- and higher-way elimination of heterogeneity from the experimental area have been described in various places in statistical literature [e.g., Fisher, 1925, 1926, 1936, 1945; Yates, 1937, 1940; Yates and Hale, 1939; Bose and Kishen, 1939; Youden, 1940; Cochran et al., 1941; Kishen, 1942; Kempthorne and Federer, 1948; Shrikhande, 1951; Stet ~~Kempthorne, 1952;~~ Pearce, 1952; Mandel, 1954; Stet ~~Federer, 1955;~~ NaNagara, 1957; Steel, 1958; Patterson and Lucas, 1959]; a large number of these have been described by Kempthorne [1952] and Federer [1955] in their textbooks. The augmentation of any of these designs with additional treatments gives rise to an augmented two-, three-, or higher-restrictional experimental design. Federer [1956a, 1956b] has presented analyses for the augmented latin square design and discusses other augmented designs of the latin square type.

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II. Augmented Designs with Two-way Elimination of Heterogeneity

Experimental designs with two-way elimination of heterogeneity are the latin square design and variations of the latin square design (e.g., the Youden square, latin squares with additional rows and/or columns, latin squares with less than the standard number of rows, columns or treatments, etc. [Federer, 1955, chs. XII and XIV, sec. XIII-5; Kempthorne, 1952, chs. 24 and 29, sec. 26.5; Shrikhande, 1951]). The construction of augmented designs from the designs in this group is illustrated with examples. Consider first the augmented latin square design. Let $v_b = b = 3$ standards and $v_1 = 16$ new treatments be arranged in a 3x3 augmented latin square design; the following design represents one possible experimental arrangement (capital letters represent standards and numbers represent new treatments):

	Column								
Row	1			2			3		
1	C	1	2	A	3	4	5	B	6
2	A	12	11	10	B	9	8	C	7
3	-	B	13	14	15	C	16	A	-

The number of new treatments in the hth row and jth column, m_{hj}, need not be constant.

The randomization procedure for the augmented latin square design follows:

(i) Select a random arrangement for the $v_b = b$ standards in a bxb latin square [Federer, 1955, ch. VI].

(ii) There are $M_{hj} = n_{hj} + 1$ experimental units falling in the hth row and jth column. Randomly allocate the standard to one of the M_{hj} units.

(iii) Randomly assign the new treatments to the remaining units with the proviso that no new treatments appear twice in a row or twice in a column until it has appeared once in each row and once in each column.

One of the possible experimental arrangements for an augmented Youden square design with $v_b = 4$ standards (capital letters) and $v_1 = 17$ new treatments (numbers) in $r = 3$ rows and $c = b = 4$ columns is given below:

	Column									
Row	1			2			3		4	
1	A	1	2	C	3	4	B	5	6	D
2	7	D	8	9	B	10	A	11	C	12
3	13	14	C	15	16	A	17	D	B	-

The randomization procedure for an augmented incomplete latin square design, such as the augmented Youden square design, follows:

(i) Randomly allot the rows and then the columns of the arrangement of the standards to the rows and columns of the experimental area.

(ii) Randomly allot the standards to one of the M_{hj} units in the hth row and jth column.

(iii) Randomly assign the new treatments to the remaining units with the proviso that no new treatments appear twice in a row or twice in a column until it has appeared once in each row and once in each column.

The construction of augmented latin square designs with additional rows and/or columns is similar to that for the designs given above.

For other augmented designs with two-way elimination of heterogeneity, a similar procedure is followed. For example, consider the augmented simple change-over design with $v_b=3$ (capital letters), $v_1=15$ (lower case letters), $r=v_b=3$, $c=kv_b=6$ (k an integer). The systematic plan for this design follows:

	Sequence					
Period (row)	1	2	3	4	5	6
1	A d	A e	B f	B g	C h	C i
2	B j	C k	A l	C m	A n	B o
3	C p	B q	C r	A -	B -	A -

The randomization procedure for the augmented simple change-over design is:

(i) Randomly allot the sequence to the column.

(ii) Randomly allot the capital letter (standard treatment) to one of the $M_{hj}=m_{hj}+1$ units in the hth row and jth column.

(iii) Randomly assign the new treatments to the remaining experimental units with the proviso that no new treatment will appear twice in a sequence or row until it has appeared once in each sequence and in each row.

The construction of, and the randomization for, an augmented tied latin square design [Federer, 1955, page 440] is identical to that for a simple change-over design.

General methods of analysis for these designs are developed in a manner similar to that given for augmented designs with one-way elimination of heterogeneity [Federer, 1961]. The general linear model is:

$$Y_{hij}=n_{hij}(\mu+\rho_h+\gamma_j+\tau_i+\epsilon_{hij}) , \qquad \text{(II-1)}$$

where $n_{hij}=1$ if the ith treatment appears in the hth row and jth column and $= 0$ otherwise, $\mu=$ a general mean effect, $\rho_h=$ effect of hth row, $\gamma_j=$ effect of jth column, $\tau_i=$ effect of ith treatment, and $\epsilon_{hij}=$ a random effect normally and independently distributed with mean zero and variance σ^2_ϵ; $h=1,2,\cdots,r$; $j=1,2,\cdots,c$; $i=1,2,\cdots,v$.

II.1 Intrablock analysis

The least squares estimates of effects ignoring interrow and intercolumn information, are obtained by minimizing the following sum of squares:

$$E=\sum_h \sum_i \sum_j n_{hij}(Y_{hij}-\mu-\rho_h-\gamma_j-\tau_i)^2 , \qquad \text{(II-2)}$$

using restrictions of the form:

$$\sum_{h=1}^{r} \hat{\rho}_h = \sum_{i=1}^{v} \hat{\tau}_i = \sum_{j=1}^{c} \hat{\gamma}_j = 0 , \qquad \text{(II-3)}$$

where $\hat{\rho}_h$, $\hat{\tau}_i$, and $\hat{\gamma}_j$, the estimates satisfying equations (II-3) to (II-7), are estimates of ρ_h, τ_i, and γ_j, respectively. The resulting normal equations for $\hat{\mu}$, $\hat{\rho}_h$, $\hat{\tau}_i$, and $\hat{\gamma}_j$, respectively, are:

$$n_{\cdot\cdot\cdot}\hat{\mu} + \sum_{h=1}^{r} n_{h\cdot\cdot}\hat{\rho}_h + \sum_{i=1}^{v} n_{\cdot i\cdot}\hat{\tau}_i + \sum_{j=1}^{c} n_{\cdot\cdot j}\hat{\gamma}_j = \Sigma\Sigma\Sigma n_{hij}Y_{hij}=Y_{\cdot\cdot\cdot} ; \qquad \text{(II-4)}$$

$$n_{h..}(\hat{\mu}+\hat{\rho}_h)+\sum_i n_{hi.}\hat{\tau}_i+\sum_j n_{h.j}\hat{\gamma}_j=\sum_{ij}\sum n_{hij}Y_{hij}=Y_{h..}\ ; \qquad \text{(II-5)}$$

$$n_{.i.}(\hat{\mu}+\hat{\tau}_i)+\sum_h n_{hi.}\hat{\rho}_h+\sum_j n_{.ij}\hat{\gamma}_j=\sum_{hj}\sum n_{hij}Y_{hij}=Y_{.i.}\ ; \qquad \text{(II-6)}$$

$$n_{..j}(\hat{\mu}+\hat{\gamma}_j)+\sum_h n_{h.j}\hat{\rho}_h+\sum_i n_{.ij}\hat{\tau}_i=\sum_{hi}\sum n_{hij}Y_{hij}=Y_{..j}\ , \qquad \text{(II-7)}$$

where $n_{h..}=\sum_{ij}\sum n_{hij}$ = number of experimental units in row h, $n_{.i.}=\sum_{hj}\sum n_{hij}$ = number of units in which treatment i appears, $n_{..j}=\sum_{hi}\sum n_{hij}$ = number of units in jth column, and $n_{...}=\sum_{hij}\sum\sum n_{hij}$ = total number of experimental units.

The solution for the unknowns from equations (II-3) to (II-7) is not simple unless the $n_{ki.}$, $n_{.ij}$, and/or $n_{h.j}$ are constant for the standards and/or for the new treatments. The kth equation from the v equations in $\hat{\tau}_i$ may be expressed as:

$$n_{.k.}\hat{\tau}_k-\sum_{h=1}^{r}\frac{n_{hk.}}{n_{h..}}\sum_{i=1}^{v}(n_{hi.}-d_k)\hat{\tau}_i=Y_{.k.}-\sum_{h=1}^{r}\frac{n_{hk.}}{n_{h..}}Y_{h..}$$

$$-\sum_{j=1}^{c}n_{.kj}\hat{\gamma}_j+\sum_{h=1}^{r}\frac{n_{hk.}}{n_{h..}}\sum_{j=1}^{r}n_{h.j}\hat{\gamma}_j \qquad \text{(II-8)}$$

or as:

$$n_{.k.}\hat{\tau}_k-\sum_{j=1}^{c}\frac{n_{.kj}}{n_{..j}}\sum_{i=1}^{v}(n_{.ij}-d_k)\hat{\tau}_i=Y_{.k.}-\sum_{j=1}^{c}\frac{n_{.kj}}{n_{..j}}Y_{..j}$$

$$-\sum_{h=1}^{r}n_{hi.}\hat{\rho}_h+\sum_{j=1}^{c}\frac{n_{.kj}}{n_{..j}}\sum_{h=1}^{r}n_{h.j}\hat{\rho}_j\ , \qquad \text{(II-9)}$$

where d_k = an arbitrary and convenient constant making use of the restrictions on the $\hat{\tau}_i$ in (II-3); at least one of the $d_k \neq 0$.

Likewise, the gth equation of the r equations involving the $\hat{\rho}_h$ is

$$n_{g..}\hat{\rho}_g - \sum_{j=1}^{c} \frac{n_{g.j}}{n_{..j}} \sum_{h=1}^{r} (n_{h.j} - d_g)\hat{\rho}_h = Y_{g..} - \sum_{j=1}^{c} \frac{n_{g.j}}{n_{..j}} Y_{..j}$$

$$- \sum_{i=1}^{v} n_{gi.}\hat{\tau}_i + \sum_{j=1}^{c} \frac{n_{g.j}}{n_{..j}} \sum_{i=1}^{v} n_{.ij}\hat{\tau}_i \; ; \qquad \text{(II-10)}$$

the fth equation of the c equations involving the $\hat{\gamma}_j$ is:

$$n_{..f}\hat{\gamma}_f - \sum_{h=1}^{r} \frac{n_{h.f}}{n_{h..}} \sum_{j=1}^{c} (n_{h.j} - d_f) = Y_{..f} - \sum_{h=1}^{r} \frac{n_{h.f}}{n_{h..}} Y_{h..}$$

$$- \sum_{i=1}^{v} n_{.if}\hat{\tau}_i + \sum_{h=1}^{r} \frac{n_{h.f}}{n_{h..}} \sum_{i=1}^{v} n_{hi.}\hat{\tau}_i \; , \qquad \text{(II-11)}$$

where d_g and d_f are arbitrary, convenient constants making use of the restrictions in (II-3).

If $r \leq c$ and if the coefficients are unequal the simplest procedure is to solve for the $\hat{\rho}_j$ from (II-10) in terms of the observations and the $\hat{\tau}_i$. These results are then substituted in (II-9); after collecting coefficients for the $\hat{\tau}_i$, a set of normal equations in the $\hat{\tau}_i$ results. These v equations in the $\hat{\tau}_i$ may then be solved to obtain the $\hat{\tau}_i$ in terms of the observations. If $r > c$ and if the coefficients are unequal then solve for the $\hat{\gamma}_j$ from (II-11) and substitute these values in (II-8) and solutions are obtained for the $\hat{\tau}_i$.

In several augmented designs with two-way elimination of heterogeneity the $n_{hi.}$, $n_{h.j}$, and/or the $n_{.ij}$ will be constant for a given i.

This simplifies the equations in that (II-8) or (II-9) or both may involve only the $\hat{\tau}_i$ and the observations. Unless the heterogeneity in the experimental area dictates otherwise, it is helpful computationally to equalize the $n_{hi\cdot}$, $n_{h\cdot j}$, and the $n_{\cdot ij}$ for each i.

The various sums of squares are:

<u>Total with $\sum_{h=1}^{r}\sum_{j=1}^{c} M_{hj}-1=n_{\cdot\cdot\cdot}-1$ d.f.:</u>

$$U-F=\sum_{hij}\sum\sum n_{hij}Y^2_{hij}-\frac{Y^2_{\cdot\cdot\cdot}}{n_{\cdot\cdot\cdot}} \; . \tag{II-12}$$

<u>Row (ignoring column and treatment) with r-1 d.f.:</u>

$$R=\sum_{h=1}^{r}\frac{Y^2_{h\cdot\cdot}}{n_{h\cdot\cdot}}-\frac{Y^2_{\cdot\cdot\cdot}}{n_{\cdot\cdot\cdot}} \; . \tag{II-13}$$

<u>Column (ignoring row and treatment) with c-1 d.f.:</u>

$$C=\sum_{j=1}^{c}\frac{Y^2_{\cdot\cdot j}}{n_{\cdot\cdot j}}-\frac{Y^2_{\cdot\cdot\cdot}}{n_{\cdot\cdot\cdot}} \; . \tag{II-14}$$

<u>Treatment (ignoring row and column) with v-1 d.f.:</u>

$$T=\sum_{i=1}^{v}\frac{Y^2_{\cdot i\cdot}}{n_{\cdot i\cdot}}-\frac{Y^2_{\cdot\cdot\cdot}}{n_{\cdot\cdot\cdot}} \; . \tag{II-15}$$

<u>Row (ignoring treatment; eliminating column) with r-1 d.f.:</u>

$$R^*=\mu^*Y_{\cdot\cdot\cdot}+\Sigma\rho_h^*Y_{h\cdot\cdot}+\sum_j\gamma_j^*Y_{\cdot\cdot j}-\sum_j\frac{Y^2_{\cdot\cdot j}}{n_{\cdot\cdot j}} \; , \tag{II-16}$$

where the starred estimates are obtained for (II-3), (II-4), (II-5), and (II-7) with each $\hat{\tau}_i$ set equal to zero.

<u>Column (ignoring treatment; eliminating row) with c-1 d.f.:</u>

$$C^{*}=\mu^{*}Y_{...}+\sum_{h}\rho_h^{*}Y_{h..}+\sum_{j}\gamma_j^{*}Y_{..j}-\sum_{h}\frac{Y_{h..}^2}{n_{h..}} \quad . \tag{II-17}$$

<u>Treatment (eliminating row and column) with v-1 d.f.:</u>

$$T'=\hat{\mu}Y_{...}+\sum_{h}\hat{\rho}_hY_{h..}+\sum_{i}\hat{\tau}_iY_{.i.}+\sum_{j}\hat{\gamma}_jY_{..j}-\mu^{*}Y_{...}$$

$$-\sum_{h}\rho_h^{*}Y_{h..}-\sum_{j}\gamma_j^{*}Y_{..j} \quad . \tag{II-18}$$

<u>Row (eliminating treatment and column) with r-1 d.f.:</u>

$$R'=\hat{\mu}Y_{...}+\sum_{h}\hat{\rho}_hY_{h..}+\sum_{i}\hat{\tau}_iY_{.i.}+\sum_{j}\hat{\gamma}_jY_{..j}-\mu'Y_{...}$$

$$-\sum_{i}\tau_i'Y_{.i.}-\sum_{j}\gamma_j'Y_{..j} \quad , \tag{II-19}$$

where the primed estimates are obtained from (II-3), (II-4), (II-6), and (II-7) with each $\hat{\rho}_h$ set equal to zero.

<u>Column (eliminating treatment and row) with c-1 d.f.:</u>

$$C'=\hat{\mu}Y_{...}+\sum_{h}\hat{\rho}_hY_{h..}+\sum_{i}\hat{\tau}_iY_{.i.}+\sum_{j}\hat{\gamma}_jY_{..j}-\mu''Y_{...}$$

$$-\sum_{h}\rho_h''Y_{h..}-\sum_{i}\tau_i''Y_{.i.} \quad , \tag{II-20}$$

where the double-primed estimates are obtained from (II-3), (II-4), (II-5), and (II-6) with each $\hat{\gamma}_j$ set equal to zero.

The error sum of squares is obtained as:

$$E_e=\sum\sum\sum_{hij} n_{hij}Y_{hij}^2-\hat{\mu}Y_{...}-\sum_{h}\hat{\rho}_hY_{h..}-\sum_{i}\hat{\tau}_iY_{.i.}-\sum_{j}\hat{\gamma}_jY_{..j}=U-R-C^{*}-T'$$

$$=U-C-R^{*}-T'=\mu-T-R^{+}-C'=U-T-C^{+}-R' \qquad \text{(II-21)}$$

where R^{+} = Row (eliminating treatment; ignoring column) sum of squares and C^{+} = column (eliminating treatment; ignoring row) sum of squares.

As explained earlier it is possible to obtain a set of v equations involving only the $\hat{\tau}_i$ on the left side and the observations on the right. The v equations may then be written in the form $N\hat{\tau}=Q$ where N is the v x v matrix of coefficients obtained from (II-8) or (II-9), $\hat{\tau}$ is a v x 1 column vector, and Q is a v x 1 column vector. Therefore, $\hat{\tau}=N^{-1}Q$ and the estimates $\hat{\tau}_i$ are expressible as a linear combination of the Q_i which are the right hand sides of the normal equations. The variance of any $\hat{\tau}_i$ or of the difference between two $\hat{\tau}_i$ may be obtained directly from the inverse matrix of coefficients in the solution of the $\hat{\tau}_i$.

II.2 Interblock analysis

The least squares estimates of effects utilizing interrow and intercolumn information are obtained by minimizing the following sum of squares:

$$w \sum_{h=1}^{r} \sum_{i=1}^{v} \sum_{j=1}^{c} n_{hij}(Y_{hij}-\mu-\rho_h-\tau_i-\gamma_j)^2$$

$$+w_r \sum_{h=1}^{r}(Y_{h..}-n_{h..}\mu-\sum_{i=1}^{v} n_{hi.}\tau_i-\sum_j n_{h.j}\gamma_j)^2/n_{h..}$$

$$+w_c \sum_{j=1}^{c}(Y_{..j}-n_{..j}\mu-\sum_{i=1}^{v} n_{.ij}\tau_i-\sum_{h=1}^{r} n_{h.j}\rho_h)^2/n_{..j}\ , \qquad \text{(II-22)}$$

where $w=1/\hat{\sigma}_\epsilon^2$, $w_r=(\hat{\sigma}_\epsilon^2+k_1\hat{\sigma}_\rho^2)^{-1}$, $w_c=(\hat{\sigma}_\epsilon^2+k_2\hat{\sigma}_\gamma^2)^{-1}$, ρ_h and γ_j are assumed to be normally and independently distributed with mean zero and variances σ_ρ^2 and σ_γ^2, respectively, and k_1 and k_2 are constants obtained from the analysis of the three-way classification with unequal numbers and are functions involving elements of the inverse matrix [see Henderson, 1953; Federer, 1960].

The normal equations for $\hat{\mu}$, $\hat{\rho}_{h=g}$, $\hat{\tau}_{i=k}$, and $\hat{\gamma}_{j=f}$ respectively, are:

$$\hat{\mu}n_{\cdot\cdot\cdot}(w+w_r+w_c)+(w+w_r+w_c)\sum_{i=1}^{v} n_{\cdot i\cdot}\hat{\tau}_i$$

$$+(w+w_c)\sum_{h=1}^{r} n_{h\cdot\cdot}\hat{\rho}_h+(w+w_r)\sum_{j=1}^{c} n_{\cdot\cdot j}\hat{\gamma}_j=Y_{\cdot\cdot\cdot}(w+w_r+w_c) \; ; \quad \text{(II-25)}$$

$$\hat{\mu}n_{g\cdot\cdot}(w+w_c)+n_{g\cdot\cdot}w\hat{\rho}_h+w_c\sum_{j=1}^{c}\frac{n_{g\cdot j}}{n_{\cdot\cdot j}}\sum_{h=1}^{r} n_{h\cdot j}\hat{\rho}_h$$

$$+w\sum_{i=1}^{v} n_{gi\cdot}\hat{\tau}_i+w_c\sum_{j=1}^{c}\frac{n_{g\cdot j}}{n_{\cdot\cdot j}}\sum_{i=1}^{v} n_{\cdot ij}\hat{\tau}_i+w\sum_{j=1}^{c} n_{g\cdot j}\hat{\gamma}_j$$

$$=wY_{g\cdot\cdot}+w_c\sum_{j=1}^{c}\frac{n_{g\cdot j}}{n_{\cdot\cdot j}}Y_{\cdot\cdot j} \; ; \quad \text{(II-26)}$$

$$\hat{\mu}n_{\cdot k\cdot}(w+w_r+w_c)+wn_{\cdot k\cdot}\hat{\tau}_k+w_r\sum_{h=1}^{r}\frac{n_{hk\cdot}}{n_{h\cdot\cdot}}\sum_{i=1}^{v} n_{hi\cdot}\hat{\tau}_i$$

$$+w_c\sum_{h=1}^{c}\frac{n_{\cdot kj}}{n_{\cdot\cdot j}}\sum_{i=1}^{v} n_{\cdot ij}\hat{\tau}_i+w\sum_{h} n_{hk\cdot}\hat{\rho}_h+w_c\sum_{j=1}^{c}\frac{n_{\cdot kj}}{n_{\cdot\cdot j}}\sum_{h=1}^{r} n_{h\cdot j}\hat{\rho}_h$$

$$+w\sum_{j=1}^{c} n_{\cdot kj}\hat{\gamma}_j+w_r\sum_{h=1}^{r}\frac{n_{hk\cdot}}{n_{h\cdot\cdot}}\sum_{j=1}^{c} n_{h\cdot j}\hat{\gamma}_j$$

$$=wY_{\cdot k\cdot}+w_r\sum_{h=1}^{r}\frac{n_{hk\cdot}}{n_{h\cdot\cdot}}Y_{h\cdot\cdot}+w_c\sum_{j=1}^{c}\frac{n_{\cdot kj}}{n_{\cdot\cdot j}}Y_{\cdot\cdot j} \; ; \quad \text{(II-27)}$$

$$\hat{\mu}n_{\cdot\cdot f}(w+w_r)+w\sum_{h=1}^{r}n_{h\cdot f}\hat{\rho}_h+w\sum_{i=1}^{v}n_{\cdot if}\hat{\tau}_i+w_r\sum_{h=1}^{r}\frac{n_{h\cdot f}}{n_{h\cdot\cdot}}\sum_{i=1}^{v}n_{hi\cdot}\hat{\tau}_i$$

$$+wn_{\cdot\cdot f}\hat{\gamma}_f+w_r\sum_{h=1}^{r}\frac{n_{h\cdot f}}{n_{h\cdot\cdot}}\sum_{j=1}^{c}n_{h\cdot j}\hat{\gamma}_j$$

$$=wY_{\cdot\cdot f}+w_r\sum_{h=1}^{r}\frac{n_{h\cdot f}}{n_{h\cdot\cdot}}Y_{h\cdot\cdot}\quad . \tag{II-28}$$

Equations (II-3) and (II-25) to (II-28) result in unique estimates for the $\hat{\mu}$, $\hat{\rho}_h$, $\hat{\tau}_i$, and $\hat{\gamma}_j$. Since different simplifications are possible for each augmented design with two-way elimination of heterogeneity, the above equations will not be manipulated further in the manner of previous sections.

The sums of squares given in section II-1 are directly applicable here. The variances for the $\hat{\tau}_i$ and differences among the $\hat{\tau}_i$ are computed from the elements of the inverse matrix in the solution of the $\hat{\tau}_i$. For example, given that $\hat{\tau}_i=\sum_{g=1} n^{gi}Q_g$, where n^{gi} are the elements of the inverse matrix and the Q_g are the right hand sides of normal equations in terms of the observations and weights w, w_r and w_c for the $\hat{\tau}_i$, the estimated variance of a difference between $\hat{\tau}_i$ and $\hat{\tau}_g$ is $V(\hat{\tau}_i-\hat{\tau}_g)=n^{ii}+n^{gg}-n^{gi}-n^{ig}$.

III. Augmented Designs with Three-way Elimination of Heterogeneity

Experimental designs with three-way elimination of heterogeneity are the magic latin square design, the latin cube design, the lattice square design[1], etc. [See Federer, 1955, chs X to XV; Kempthorne, 1952, chs. 19 and 24] The augmented designs in this group are constructed by augmenting the original design with new treatments. For example, consider the augmented magic latin square for $v_b = 4$ standard treatments (A, B, C, D) replicated 4 times and $v_1 = 9$ new treatments (e, f, g, h, i, j, k, l, m) replicated once. One possible arrangement is:

	Column number			
Row number	1	2	3	4
1	A f	e B	C g	D h
2	C	D	A	B
3	D i	C j	k B	A l
4	B	A	D	m C

The square numbers are

1	2
3	4

, and as indicated above, the number of new treatments in a row, a column, or a square need not be constant.

The randomization procedure for the augmented magic latin square follows:

(i) Select at random one of the possible arrangements for a magic latin square (e.g., follow the procedure described on p. 142 of Federer [1955] except to add the word square in step (iii). Then, assign letters to treatments at random).

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1 In one sense this design is a two-restrictional design while in another sense it is a three-restrictional design as considered here.

(ii) There are $n_{gh.j}$ experimental units falling in the gth row, the hth column, and the jth square. Randomly allocate the standard treatment in this category to one of the $n_{gh.j}$ units.

(iii) Randomly allocate the new treatments to the remaining experimental units.

(iv) If a new treatment is included more than once, randomly allocate the remaining entries of the treatment with the provisos that no treatment is to be included twice in a row, column, or square until it has occurred once in each row, column, and square.

As a second example consider the 3 x 3 semi-balanced lattice square augmented with 17 treatments (numbers). The schematic arrangement follows (standards designated by capital letters and dashes put in at random):

I

A 1	B 2	C 3
D 4	E 5	F 6
G 7	H 8	I 9

II

A 10	F -	H 11
E 12	G 13	C 14
I 15	B 16	D 17

Adding the following two arrangements and 15 additional new treatments results in an augmented balanced lattice design:

III

A -	D 18	G 19
B -	E 20	H 21
C 22	F 23	I 24

IV

A 25	E 26	I 27
F 28	G -	B 29
H 30	C 31	D 32

As indicated, the number of experimental units, $n_{gh.j}$, in the hth row and jth column of the gth replicate need not be constant. The analysis is simpler if $n_{gh.j}$ is a constant.

The randomization procedure follows:

(i) Select a random arrangement of a lattice square design for k^2 standards as described in the literature [e.g., Federer, 1955, page 379].

(ii) Randomly allot the given standard to one of the $n_{gh.j}$ experimental units in the hth row and jth column of the gth replicate.

(iii) Randomly allocate the new treatments to the remaining $n_{...}-rk^2$ experimental units.

(iv) If a new treatment is included more than once, randomly allocate the remaining entries of the new treatment with the provisos that no new treatment is to appear in a replicate twice until it has occurred once in all replicates and no new treatment is to occur twice in the hth row or jth column until it has occurred once in all rows and columns of the gth replicate.

One of the possible arrangements for an augmented latin cube design of first order [see Federer, 1955, page 470] with $v_b = 3$ standards (A, B, C) and $v_1 = 22$ new treatments (1, 2, 3, ..., 22), follows:

I			II			III		
A	2	3	B	C	A	7	8	B
1	B	C	4	5	6	C	A	9
B	C	A	12	-	13	A	-	C
10	-	11	C	A	B	14	B	15
C	17	18	A	19	C	-	C	22
16	A	B	-	B	20	B	21	A

The number of experimental units, $n_{gh.j}$, in the hth row and jth column of the gth group need not be constant but the analysis is simpler if $n_{gh.j}$ is a constant.

The randomization procedure for the augmented latin cube design follows:

(i) Select a random arrangement of a latin cube design (e.g., use a procedure similar to that described for the magic latin square).

(ii) Randomly allot the standard treatment to one of the $n_{gh.j}$ experimental units in the hth row and jth column of the gth group.

(iii) Randomly allot the new treatments to the remaining $n_{...}-k^3$ experimental units with similar provisos to those listed for the previous two designs.

Other augmented designs with three-way elimination of heterogeneity may be constructed as described in the above three examples.

The general methods of analysis for three-restrictional augmented designs are developed along lines similar to those with two-way elimination of heterogeneity [see Henderson, 1953; Federer, 1957]. The linear model is:

$$Y_{ghij} = n_{ghij}(\mu+\tau_i+\lambda_g+\rho_h+\gamma_j+\epsilon_{ghij}), \qquad \text{(III-1)}$$

where $n_{ghij} = 1$ if ith treatment occurs in the gth group, hth row, and jth column (If the hth row and jth column is within the gth group, then read ρ_{gh} and γ_{gj} in (III-1) above and $n_{ghij} = 1$ if ith treatment occurs in hth row and jth column of the gth group) and zero otherwise, μ = a general mean effect, τ_i = effect of ith treatment, λ_g = effect of gth group, ρ_h = effect of hth row, and γ_j = effect of jth column, and ϵ_{ghij} = a random effect normally and independently distributed with mean zero and variance σ^2_ϵ; $g = 1, 2, \ldots, b$; $h = 1, 2, \ldots, r$; $i = 1, 2, \ldots, v$; $j = 1, 2, \ldots, c$.

III. 1 Intrablock analysis

Minimization of the following sum of squares,

$$E = \sum_g \sum_h \sum_i \sum_j n_{ghij}(Y_{ghij}-\mu-\tau_i-\lambda_g-\rho_h-\gamma_j)^2, \qquad \text{(III-2)}$$

with respect to the parameters results in a set of normal equations. These equations plus a set of restrictions of the form,

$$\Sigma\hat{\tau}_i = \Sigma\hat{\lambda}_g = \Sigma\hat{\rho}_h = \Sigma\hat{\gamma}_j = 0 \qquad \text{(III-3)}$$

result in unique solutions for the effects.

For designs like the lattice square, the sum of squares to be minimized is

$$\sum_{i=1}^{v} \sum_{g=1}^{b} \sum_{h=1}^{r_g} \sum_{j=1}^{c_g} n_{ghij} (Y_{ghij}-\mu-\tau_i-\lambda_g-\rho_{gh}-\gamma_{gj})^2 \qquad \text{(III-4)}$$

and the restrictions are of the form

$$\sum_{i=1}^{v} \hat{\tau}_i = \sum_{g=1}^{b} \hat{\lambda}_g = \sum_{h=1}^{r_g} \hat{\rho}_{gh} = \sum_{j=1}^{c_g} \hat{\gamma}_{gj} = 0, \qquad \text{(III-5)}$$

where r_g = number of rows in the gth group and c_g = number of columns in the gth group ($r_g = c_g = k$ in the lattice square).

The solution of the $\hat{\tau}_i$ in matrix form is the same form as described for the intrablock analysis of section II. The variances are computed similarly.

The analysis of variance, using standard regression notation, is of the form [e.g., Federer, 1957]:

Source of variation	df	ss	ms
Total (elim. mean; ign. others)	$n_{....}-1$	$\sum\limits_{ghij}\sum\sum\sum Y^2_{ghij} - \frac{Y^2_{....}}{n_{....}}$	-
Group (elim. mean; ign. others)	b-1	$\sum\frac{Y^2_{g...}}{n_{g...}} - \frac{Y^2_{....}}{n_{....}} = G - \frac{Y^2_{....}}{n_{....}}$	-
Row (elim. mean and group; ign. others)	f_r	$SS(\mu',\lambda',\rho') - G = R' - G$	-
Column (elim mean, group, row; ign. tr.)	f_c	$SS(\mu'',\lambda'',\rho'',\gamma'') - R' = C'' - R'$	-
Treatment (elim. all other effects)	v-1	$SS(\hat{\mu},\hat{\lambda},\hat{\rho},\hat{\gamma},\hat{\tau}) - C'' = \hat{T} - C''$	-
Error	$n_{...}-b-f_r-f_c-v+1$	$\sum\sum\sum\sum Y^2_{ghij}-\hat{T}$	E_e
Columns (eliminating other effects)	f_r	$\hat{T} - SS(\mu^+,\lambda^+,\rho^+,\tau^+)$	E_c
Rows (eliminating other effects)	f_c	$\hat{T} - SS(\mu^-,\lambda^-,\gamma^-,\tau^-)$	E_r
Groups (eliminating other effects)	b-1	$\hat{T} - SS(\mu^*,\rho^*,\gamma^*,\tau^*)$	E_s

III - 2. Interblock analysis

For the lattice square design with recovery of interrow and intercolumn information to adjust treatment means, the following sum of squares is minimized to obtain the normal equations [Yates, 1940; Kempthorne and Federer, 1948; Na Nagara, 1957]:

$$w \sum_{ghij}\sum\sum\sum n_{ghij}(Y_{ghij}-\mu-\lambda_g-\rho_{gh}-\gamma_{gj}-\tau_i)^2$$

$$+ w_c \sum_{gj}\sum(Y_{g..j}-n_{g.m.}(\mu+\lambda_g) - \sum_i n_{g.ij}\tau_i$$

$$- \sum_h n_{gh.j}\rho_{gh})^2 / n_{g..j}$$

$$+ w_r \sum_{gh}\sum(Y_{gh..}-n_{gh..}(\mu+\lambda_g) - \sum_i n_{ghi.}\tau_i - \sum_j n_{gh.j}\gamma_{gj})^2 / n_{gh..} \qquad \text{(III-6)}$$

For three-restrictional designs where the categories are not nested the sum of square to be minimized is:

$$w \sum_{g=1}^{b} \sum_{h=1}^{r} \sum_{i=1}^{v} \sum_{j=1}^{c} n_{ghij}(Y_{ghij}-\mu-\lambda_g-\rho_h-\gamma_j-\tau_i)^2$$

$$+ w_{er} \sum_{g=1}^{b} (Y_{g...}-n_{g...}\mu - \sum_i n_{g.i.}\tau_i - \sum_h n_{gh..}\rho_h$$

w_b

$$- \sum_j n_{g..j}\gamma_j)^2 / n_{g...}$$

$$+ w_r \sum_{h=1}^{r} (Y_{.h..}-n_{.h..}\mu - \sum_i n_{.hi.}\tau_i - \sum_g n_{gh..}\lambda_g$$

$$- \sum_j n_{.h.j}\gamma_j)^2 / n_{.h..}$$

$$+ w_c \sum_{j=1}^{c} (Y_{...j}-n_{...j}\mu - \sum_i n_{..ij}\tau_i - \sum_g n_{g..j}\lambda_g$$

$$- \sum_h n_{.h.j}\rho_h)^2 / n_{...j} \qquad \text{(III-7)}$$

The resulting normal equations plus an appropriate set of restrictions on the solutions result in unique solutions for the effects. Linear combinations of E_e, E_r, and E_c are used to compute $w = 1/E_e$, $w_r = (\hat{\sigma}^2_\epsilon + k_3\hat{\sigma}^2_\rho)^{-1}$ and $w_c = (\hat{\sigma}^2_\epsilon + k_4\hat{\sigma}^2_\gamma)^{-1}$ in formula (III-6); linear combinations of E_e, E_r, E_c, and E_s are used to compute $w = 1/E_e$, $w_r = (\hat{\sigma}^2_\epsilon + k_5\hat{\sigma}^2_\rho)^{-1}$, $w_c = \hat{\sigma}^2_\epsilon + k_6\hat{\sigma}^2_\gamma$ and $w_s = (\hat{\sigma}^2_\epsilon + k_7\hat{\sigma}^2_\lambda)^{-1}$. The expectations of E_r, E_c, and E_s, where λ, ρ, and γ are considered to be independent random variates distributed with zero mean and variances σ^2_λ, σ^2_ρ, and σ^2_γ, respectively, are obtained in the usual manner for unbalanced classifications[Henderson, 1953; Federer, 1960]. Here again the solution for the treatment effects in matrix form is $\tau^*_i = \sum_f n^{fi} Q^*_{\dots f}$, where the $Q^*_{\dots f}$ are the right hand sides of the normal equations for the τ^*_i after eliminating all other effects. The estimated variance of a difference between τ^*_i and τ^*_f $V(\tau^*_i - \tau^*_f) = n^{ii} + n^{ff} - n^{if} - n^{fi}$.

IV. Augmented Designs with Four- and higher-way Elimination of Heterogeneity.

Designs with four-or higher-way elimination of heterogeneity have been suggested and sometimes constructed[e.g., Fisher, 1936; Yates, 1937; Kempthorne and Federer, 1948; Kempthorne, 1952; Federer, 1955] but detailed results are not given for most designs. The hyper-graeco latin squares have received detailed discussions but resolvable incomplete block designs with the incomplete blocks arranged in a lattice squares arrangement for the "whole plots" and in a lattice square arrangement for the "split plots" have only been suggested [Yates, 1937]. A latin square arrangement of the "whole plots" with a latin square arrangement of the "split plots" has received limited attention. Practical application of these designs appears unlikely but this is insufficient reason for not considering this group of designs. Any of these designs may be augmented with additional new treatments.

The general analysis, both intrablock and interblock, follows that described in the previous two sections. Here again, the nearer to balance the simpler the solutions for the effects. The adjusted treatment means (or effects) and variances of effects are obtained in a manner similar to that described above.

V. Summary

Augmented designs are standard ones to which additional treatments have been added. The additional treatments may or may not be replicated the same number of times as the treatments in the standard design or the same number of times as other new treatments. The groupings of treatments into rows, columns, complete blocks, etc. may be such that the size of the group is or is not equal. Construction procedures, randomization procedures and the analyses in general form have been presented for augmented experimental designs with two-, three- and higher-way elimination of heterogeneity.

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